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Decay estimate of strong solutions to the compressible Navier-Stokes equations in critical spaces

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1 Introduction

In this article we give a summary of recent results on the stability of the compressible Navier-Stokes equation in critical spaces $\dot{B}_{2,1}^{\frac{n}{2}}\times\dot{B}_{2,1}^{\frac{n}{2}-1}$. We consider the initial value problem for the compressible Navier-Stokes equation in $\mathbb{R}^{n}$

$$\begin{cases} \partial_{t}\rho+\nabla\cdot(\rho u)=0, \\ \partial_{t}u+(u\cdot\nabla)u+\frac{\nabla P(\rho)}{\rho}=\frac{\mu}{\rho}\triangle u+\frac{\mu+\mu'}{\rho}\nabla(\nabla\cdot u), \\ (\rho,u)(0,x)=(\rho_{0},u_{0})(x). \end{cases} \tag{1}$$

Here $t>0$, $x=(x_{1},x_{2},\cdots,x_{n})\in \mathbb{R}^{n}$; the unknown functions $\rho=\rho(t,x)>0$ and $u=u(t,x)=(u_{1}(t,x),u_{2}(t,x),\cdots,u_{n}(t,x))$ denote the density and velocity, respectively; $P=P(\rho)$ is the pressure that is assumed to be a function of the density ρ; μ and μ' are the viscosity coefficients satisfying the conditions $\mu>0$ and $\mu'+2\mu>0$; and $\nabla\cdot$, ∇ and $\triangle$ denote the usual divergence, gradient and Laplacian with respect to x, respectively.

We assume that $P(\rho)$ is smooth in a neighborhood of $\bar{\rho}$ with $P'(\bar{\rho})>0$, where $\bar{\rho}$ is a given positive constant.

We derive the convergence rate of solutions of problem (1) to the constant stationary solution $(\bar{\rho},0)$ as $t\rightarrow\infty$ when the initial perturbation $(\rho_{0}-\bar{\rho},u_{0})$ is sufficiently small in critical spaces $\dot{B}_{2,1}^{\frac{n}{2}}\times\dot{B}_{2,1}^{\frac{n}{2}-1}$ and $\dot{B}_{1,\infty}^{0}$.

Matsumura-Nishida [9] showed the global in time existence of the solution of (1) for $n=3$, provided that the initial perturbation $(\rho_{0}-\bar{\rho},u_{0})$ is sufficiently small in $H^{3}(\mathbb{R}^{3})\cap L^{1}(\mathbb{R}^{3})$. Furthermore, the following decay estimates were obtained in [9]

$$\|\nabla^{k}(\rho-\bar{\rho},u)(t)\|_{L^{2}}\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\quad k=0,1. \tag{2}$$

On the other hand, Kawashita [7] showed the global existence of solutions for initial perturbations sufficiently small in $H^{s_{0}}(\mathbb{R}^{n})$ with $s_{0}=[\frac{n}{2}]+1$, $n\geq 2$. (Note that $s_{0}=2$ for $n=3$). Wang-Tan [14] then considered the case $n=3$ when the initial perturbation $(\rho_{0}-\bar{\rho},u_{0})$ is sufficiently small in $H^{2}(\mathbb{R}^{3})\cap L^{1}(\mathbb{R}^{3})$, and proved the decay estimates (2). Okita [11] showed that if $n \geq 2$ then the following estimates hold for the solution (ρ, u) of (1) :

$$\|\nabla^k(\rho-\bar{\rho},u)(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \quad k=0,\cdots,s_0,$$

provided that $(\rho_0-\bar{\rho},u_0)$ is sufficiently small in $H^{s_0}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ with $s_0=[\frac{n}{2}]+1$.

Danchin [2] proved the global existence in a critical homogeneous Besov space, i.e., a scaling invariant Besov space. The system $(1)_1-(1)_2$ is invariant under the following transformation

$$\rho_\lambda(t,x):=\rho(\lambda^2 t,\lambda x), \quad u_\lambda(t,x):=\lambda u(\lambda^2 t,\lambda x).$$

More precisely, if (ρ,u) solves (1), so dose (ρ_λ,u_λ) provided that the pressure law P has been changed into $\lambda^2 P$. Usually, we call that a functional space is a critical space for (1) if the associated norm is invariant under the transformation $(\rho,u)\to(\rho_\lambda,u_\lambda)$ (up to a constant independent of λ). Homogeneous Besov space $C([0,\infty);\dot{B}^{\frac{n}{p}}_{p,1}\times \dot{B}^{\frac{n}{p}-1}_{p,1})$ is a critical space for (1); and Danchin [2] proved the global existence in $C([0,\infty);\dot{B}^{\frac{n}{p}}_{p,1})\times\big(C([0,\infty);\dot{B}^{\frac{n}{p}-1}_{p,1})\cap L^1(0,\infty;\dot{B}^{\frac{n}{p}+1}_{p,1})\big)$ and the estimate

$$\begin{aligned}
&\sup_{t\geq 0}\{\|\rho(t)-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|u(t)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\}+\int_0^\infty \|u\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}dt\\
\leq\ & M\big(\|\rho_0-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1}\cap\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big),
\end{aligned} \tag{3}$$

if the initial perturbation is sufficiently small in $(\dot{B}^{\frac{n}{2}}_{2,1}\cap\dot{B}^{\frac{n}{2}-1}_{2,1})\times\dot{B}^{\frac{n}{2}-1}_{2,1}$ for $n\geq 2$.

On the other hand, Haspot [5] proved the local solvability in a nonhomogeneous Besov space $B^{\frac{n}{2}}_{2,1}\times B^{\frac{n}{2}-1}_{2,1}$.

Our main result gives the optimal decay rate for strong solutions in critical Besov spaces, which is stated as follows.

2 Main Results

Theorem 2.1 ([12, 13]). *Let $n\geq 2$. Then there exists $\epsilon>0$ such that if*

$$u_0\in\dot{B}^{\frac{n}{2}}_{2,1}\cap\dot{B}^0_{1,\infty},\quad (\rho_0-\bar{\rho})\in\dot{B}^{\frac{n}{2}-1}_{2,1}\cap\dot{B}^0_{1,\infty}$$

and

$$\|\rho_0-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1}\cap\dot{B}^0_{1,\infty}}+\|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap\dot{B}^0_{1,\infty}}\leq\epsilon,$$

then problem (1) *has a unique global solution* (ρ,u) *satisfying*

$$(\rho-\bar{\rho},u)\in C([0,\infty);B^{\frac{n}{2}}_{2,1})\times\big(C([0,\infty);B^{\frac{n}{2}-1}_{2,1})\cap L^1(0,\infty;\dot{B}^{\frac{n}{2}+1}_{2,1})\big).$$

Furthermore, there exists a constant $C_0>0$ such that the estimates

$$\begin{aligned}
\|(\rho-\bar{\rho},u)(t)\|_{L^2}&\leq C_0(1+t)^{-\frac{n}{4}},\\
\|(\rho-\bar{\rho},u)(t)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}&\leq C_0(1+t)^{-\frac{n}{2}+\frac{1}{2}},\\
\|(\rho-\bar{\rho})(t)\|_{\dot{B}^{\frac{n}{2}}_{2,1}}&\leq C_0(1+t)^{-\frac{n}{2}},
\end{aligned}$$

hold for $t\geq 0$.

3 Preliminaries

In this section we first introduce the notation which will be used throughout this paper. We then introduce Besov spaces, some properties of Besov spaces and usuful lemma.

3.1 Notation

Let $L^p(1 \le p \le \infty)$ denote the usual L^p-Lebesgue space on $\mathbb{R}^n$. For a nonnegative integer m, we denote by H^m the usual L^2-Sobolev space of order m. $\mathcal{S}'$ denotes dual space of the Schwartz space. The inner-product of L^2 is denoted by $(\cdot,\cdot)$. If S is any nonempty subset of $\mathbb{Z}$, sequence space $l^p(S)$ denote the usual l^p sequence space on S.

For any integer $l \ge 0, \nabla^l f$ denotes all of l-th derivatives of f.

For a function f, we denote its Fourier transform by $\mathfrak{F}[f] = \hat{f}$:

$$\mathfrak{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \quad (\xi \in \mathbb{R}).$$

The inverse Fourier transform is denoted by $\mathfrak{F}^{-1}[f] = \check{f}$,

$$\mathfrak{F}^{-1}[f](x) = \check{f}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} f(\xi)e^{i\xi\cdot x}d\xi \quad (x \in \mathbb{R}).$$

3.2 Besov spaces

Let us now define the homogeneous and nonhomogeneous Besov spaces. First we introduce the dyadic partition of unity. We can use for instance any $\{\phi, \chi\} \in C^\infty$, such that

$$\operatorname{Supp}\phi \subset \{\xi \in \mathbb{R}^n | \frac{3}{4} \le |\xi| \le \frac{8}{3}\},$$

$$\operatorname{Supp}\chi \subset \{\xi \in \mathbb{R}^n | |\xi| \le \frac{4}{3}\},$$

$$\chi(\xi) + \sum_{j\ge 0}\phi(2^{-j}\xi) = 1 \ \ for\ \xi \in \mathbb{R}^n,$$

$$\sum_{j\in\mathbb{Z}}\phi(2^{-j}\xi) = 1 \quad for\ \xi \in \mathbb{R}^n\backslash\{0\},$$

$$\operatorname{Supp}\phi(2^{-j}\cdot) \cap \operatorname{Supp}\phi(2^{-j'}\cdot) = \emptyset \quad for\ |j-j'| \ge 2,$$

$$\operatorname{Supp}\chi \cap \operatorname{Supp}\phi(2^{-j}\cdot) = \emptyset \quad for\ j \ge 1.$$

Denoting $h = \mathfrak{F}^{-1}\phi$ and $\tilde{h} = \mathfrak{F}^{-1}\chi$, we then define the dyadic blocks by

$$\begin{aligned}
\triangle_{-1}u &= \tilde{h} * u,\\
\triangle_j u &= 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)dy \quad \text{if } j \geq 0,\\
\dot{\triangle}_j u &= 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)dy \quad \text{if } j \in \mathbb{Z}.
\end{aligned}$$

The low-frequency cut-off operators are defined by

$$S_j u = \sum_{-1\leq k\leq j-1} \triangle_k u, \quad \dot{S}_j u = \sum_{k\leq j-1} \dot{\triangle}_k u.$$

Obviously we can write that: $Id = \sum_j \triangle_j$. The high-frequency cut-off operators $\tilde{S}_j$ are defined by

$$\tilde{S}_j u = \sum_{k\geq j} \dot{\triangle}_k u.$$

We define ϕ_j by $\phi_j(\xi) = \phi(2^{-j}\xi)$.

To begin with, we define Besov spaces.

Definition 1. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, and $u \in \mathcal{S}'$ we set

$$\|u\|_{B^s_{p,r}} := \left\|2^{js}\|\triangle_j u\|_{L^p}\right\|_{l^r(\{j\geq -1\})},$$

$$\|u\|_{\dot{B}^s_{p,r}} := \left\|2^{js}\|\dot{\triangle}_j u\|_{L^p}\right\|_{l^r(\mathbb{Z})}.$$

The nonhomogeneous Besov space $B^s_{p,r}$ and the homogeneous Besov space $\dot{B}^s_{p,r}$ are the sets of functions $u \in \mathcal{S}'$ such that $\|u\|_{B^s_{p,r}}$ and $\|u\|_{\dot{B}^s_{p,r}} < \infty$ respectively.

Let us state some basic lemmas for Besov spaces.

Lemma 3.1. *The following inequalities hold:*

(i) $\|\nabla\triangle_{-1}u\|_{L^2} \leq C\|\triangle_{-1}u\|_{L^2}$.

(ii) $C^{-1}2^j\|\dot{\triangle}_j u\|_{L^2} \leq \|\nabla\dot{\triangle}_j u\|_{L^2} \leq C2^j\|\dot{\triangle}_j u\|_{L^2} \quad (j \in \mathbb{Z})$.

(iii) $\|\nabla S_j u\|_{L^2} \leq C2^j\|S_j u\|_{L^2} \quad (j \geq 0)$.

(iv) $\|\tilde{S}_j u\|_{L^2} \leq C2^{-j}\|\nabla\tilde{S}_j u\|_{L^2} \quad (j \geq 0)$.

Lemma 3.1 easily follows from the Plancherel theorem.

Remark 3.2. *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we have*

(i) $C^{-1}\left(\sum_{k\leq j-1} 2^{srk}\|\dot{\triangle}u\|^r_{L^p}\right)^{\frac{1}{r}} \leq \|\dot{S}_j u\|_{\dot{B}^s_{p,r}} \leq C\left(\sum_{k\leq j-1} 2^{srk}\|\dot{\triangle}u\|^r_{L^p}\right)^{\frac{1}{r}}$

(ii) $C^{-1}\left(\sum_{k\geq j} 2^{srk}\|\dot{\triangle}u\|^r_{L^p}\right)^{\frac{1}{r}} \leq \|\tilde{S}_j u\|_{\dot{B}^s_{p,r}} \leq C\left(\sum_{k\geq j} 2^{srk}\|\dot{\triangle}u\|^r_{L^p}\right)^{\frac{1}{r}}$

One can easily prove Remark 3.2.

Lemma 3.3. *The following properties hold:*

(*i*) $C^{-1}\|u\|_{\dot{B}^s_{p,r}} \leq \|\nabla u\|_{\dot{B}^{s-1}_{p,r}} \leq C\|u\|_{\dot{B}^s_{p,r}}$.

(*ii*) $\|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}$.

(*iii*) *If* $s' > s$ *or if* $s' = s$ *and* $r_1 \leq r$ *then* $B^{s'}_{p,r_1} \subset B^s_{p,r}$.

(*iv*) *If* $r_1 \leq r$ *then* $\dot{B}^s_{p,r_1} \subset \dot{B}^s_{p,r}$.

(*v*) *Let* $\Lambda := \sqrt{-\Delta}$ *and* $t \in \mathbb{R}$. *Then the operator* Λ^t *is an isomorphism from* $\dot{B}^s_{2,1}$ *to* $\dot{B}^{s-t}_{2,1}$.

See, e.g., [2], [3] and [5] for a proof of Lemma 3.3.

Lemma 3.4. *The following properties hold:*

(*i*) $\|u\|_{L^\infty} \leq C\|u\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \qquad \left(\dot{B}^{\frac{n}{2}}_{2,1} \subset L^\infty\right)$.

(*ii*) $\dot{B}^0_{1,1} \subset L^1 \subset \dot{B}^0_{1,\infty}$.

(*iii*) $B^s_{2,2} = H^s$.

(*iv*) $B^s_{p,r} \subset \dot{B}^s_{p,r}$ $(s > 0)$.

See, e.g., [2], [3] and [5] for a proof of Lemma 3.4.

Lemma 3.5. *Let* $1 \leq p \leq q \leq \infty$. *Assume that* $f \in L^p(\mathbb{R}^n)$. *Then for any* $\alpha \in (\mathbb{N} \cup \{0\})^n$, *there exist constants* C_1, C_2 *independent of* f, j *such that*

$$\operatorname{Supp} \hat{f} \subseteq \{|\xi| \leq A_0 2^j\} \Longrightarrow \|\partial_x^\alpha f\|_{L^q} \leq C_1 2^{j|\alpha| + jn(\frac{1}{p} - \frac{1}{q})}\|f\|_{L^p},$$

$$\operatorname{Supp} \hat{f} \subseteq \{A_1 2^j \leq |\xi| \leq A_2 2^j\} \Longrightarrow \|f\|_{L^p} \leq C_2 2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial_x^\beta f\|_{L^p}.$$

See, e.g., [1] for a proof of Lemma 3.5.

We next state some basic lemmas.

Lemma 3.6. *Let* $s_1, s_2 \leq \frac{n}{2}$ *such that* $s_1 + s_2 > 0$; *and let* $u \in \dot{B}^{s_1}_{2,1}$ *and* $v \in \dot{B}^{s_2}_{2,1}$. *Then* $uv \in \dot{B}^{s_1+s_2-\frac{n}{2}}_{2,1}$ *and*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{n}{2}}_{2,1}} \leq C\|u\|_{\dot{B}^{s_1}_{2,1}}\|v\|_{\dot{B}^{s_2}_{2,1}}.$$

See, e.g., [1], for a proof of Lemma 3.6.

Lemma 3.7. *Let* $s > 0$ *and let* $u \in \dot{B}^s_{2,1} \cap L^\infty$. *Let* $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^n)$ *such that* $F(0) = 0$. *Then* $F(u) \in \dot{B}^s_{2,1}$. *Moreover, there exists a function* C_1 *of one variable depending only on* s, n *and* F *such that*

$$\|F(u)\|_{\dot{B}^s_{2,1}} \leq C_1(\|u\|_{L^\infty})\|u\|_{\dot{B}^s_{2,1}}.$$

See, e.g., [2], for a proof of Lemma 3.7.

Lemma 3.8. *(i) Let $a, b > 0$ satisfying $\max\{a,b\} > 1$. Then*

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b}ds \le C(1+t)^{-\min\{a,b\}}, \quad t \ge 0.$$

(ii) Let $f \in L^p(0,\infty)$ and $a, b > 0$ satisfying $\max\{a,b\} > \frac{1}{p'}$ for $1 \le p \le \infty$ and p' is the conjugate exponent to p. Then

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b} f ds \le C(1+t)^{-\min\{a,b\}} \Big(\int_0^t |f|^p ds \Big)^{\frac{1}{p}}, \quad t \ge 0.$$

For a proof of (i), see [10]. Proof of (ii) is given by using Hölder inequality; we omit it.

Let us now introduce a few bilinear estimates in Besov spaces. We will use the Bony decomposition

$$uv = T_u v + T_v u + R(u,v), \tag{4}$$

with

$$T_u v = \sum_{j\in\mathbb{Z}} \dot S_{j-1} u \dot\Delta_j v, \quad R(u,v) = \sum_{j\in\mathbb{Z}} \dot\Delta_j u \tilde\Delta_j v, \quad \tilde\Delta_j v = \dot\Delta_{j-1} v + \dot\Delta_j v + \dot\Delta_{j+1} v.$$

Lemma 3.9. *It holds that*

(i)

$$\sup_{j<0} \|\dot\Delta uv\|_{L^1} \le C(\|\dot S_4 u\|_{L^2}\|\dot S_4 v\|_{L^2} + \|\tilde S_0 u\|_{L^2}\|\tilde S_0 v\|_{L^2}).$$

(ii) If $0 \le s_1, s_2, s_3, s_4 \le \frac{n}{2}$, then

$$\begin{aligned}\sum_{j\ge 0} 2^{s_1 j}\|\Delta_j uv\|_{L^2} \;&\le\; C\big(\|\dot S_{-5} u\|_{\dot B^{\frac{n}{2}-s_2}_{2,1}} \|\tilde S_{-5} v\|_{\dot B^{s_1+s_2}_{2,1}} + \|\dot S_{-5} v\|_{\dot B^{\frac{n}{2}-s_3}_{2,1}} \|\tilde S_{-5} v\|_{\dot B^{s_1+s_3}_{2,1}} \\ &\quad + \|\tilde S_{-5} u\|_{\dot B^{\frac{n}{2}-s_4}_{2,1}} \|\tilde S_{-5} v\|_{\dot B^{s_1+s_4}_{2,1}}\big).\end{aligned}$$

Proof of Lemma 3.9. We have

$$\dot\Delta_j T_g f = \sum_{|j'-j|\le 4} \dot\Delta_j(\dot S_{j'-1} g \dot\Delta_{j'} f), \quad \dot\Delta_j R(f,g) = \sum_{j'\ge j-3} \dot\Delta_j(\dot\Delta_{j'} f \tilde\Delta_{j'} g).$$

For any $j < 0$, by the Hölder inequality, we have

$$\begin{aligned}\|\dot\Delta_j T_g f\|_{L^1} \;&\le\; C \sum_{|j'-j|\le 4} \|\dot S_{j'-1} g \dot\Delta_{j'} f\|_{L^1} \\ &\le\; C\|\dot S_4 g\|_{L^2}\|\dot S_4 f\|_{L^2},\end{aligned}$$

and

$$
\begin{aligned}
\|\dot{\triangle}_j R(f,g)\|_{L^1} &\le C\|\sum_{j'\ge j-3}\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^1}\\
&\le C\sum_{j'\le 0}\|\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g\|_{L^1}+\sum_{j'\ge 1}\|\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g\|_{L^1}\\
&\le C(\|\dot{S}_3 f\|_{L^2}\|\dot{S}_3 g\|_{L^2}+\|\tilde{S}_0 f\|_{L^2}\|\tilde{S}_0 g\|_{L^2}).
\end{aligned}
$$

Taking the supremum in $j<0$, we obtain the desired estimates of (i).

We next prove (ii). Choose $s_1\in[0,\frac{n}{2}]$. We then obtain by Hölder inequality and Lemma 3.5 that

$$
\begin{aligned}
\sum_{j\ge 0}2^{s_1 j}\|\dot{\triangle}_j T_g f\|_{L^2} &\le C\sum_{j\ge 0}\sum_{|j'-j|\le 4}2^{s_1 j}\|\dot{\triangle}_j(\dot{S}_{j'-1}g\dot{\triangle}_{j'}f)\|_{L^2}\\
&\le C\sum_{j'\ge -4}2^{s_1 j'}\|\dot{S}_{j'-1}g\dot{\triangle}_{j'}f\|_{L^2}\\
&\le C\sum_{j'\ge -4}2^{s_1 j'}\|\{\dot{S}_{-5}g+(\dot{S}_{j'-1}-\dot{S}_{-5})g\}\dot{\triangle}_{j'}f\|_{L^2}\\
&\le C\sum_{j'\ge -4}2^{s_1 j'}\{\|\dot{S}_{-5}g\|_{L^{\frac{n}{s_2}}}\|\dot{\triangle}_{j'}f\|_{L^{\frac{2n}{n-2s_2}}}\\
&\quad+\|(\dot{S}_{j'-1}-\dot{S}_{-5})g\|_{L^{\frac{n}{s_3}}}\|\dot{\triangle}_{j'}f\|_{L^{\frac{2n}{n-2s_3}}}\}\\
&\le C(\|\dot{S}_{-5}g\|_{\dot{B}^{\frac{n}{2}-s_2}_{2,1}}\|\tilde{S}_{-5}g\|_{\dot{B}^{s_1+s_2}_{2,1}}+\|\tilde{S}_{-5}g\|_{\dot{B}^{\frac{n}{2}-s_3}_{2,1}}\|\tilde{S}_{-5}g\|_{\dot{B}^{s_1+s_3}_{2,1}}),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j\ge 0}2^{s_1 j}\|\triangle_j R(f,g)\|_{L^2} &\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{s_1 j}\|\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^2}\\
&\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{(s_1+\frac{n}{2})j}\|\dot{\triangle}_j(\dot{\triangle}_{j'}f\tilde{\triangle}_{j'}g)\|_{L^1}\\
&\le C\sum_{j\ge 0}\sum_{j'\ge j-3}2^{(s_1+\frac{n}{2})(j-j')}2^{(\frac{n}{2}-s_4)j'}\|\dot{\triangle}_{j'}f\|_{L^2}2^{(s_1+s_4)j'}\|\tilde{\triangle}_{j'}g\|_{L^2}\\
&\le C\|\tilde{S}_{-4}f\|_{\dot{B}^{\frac{n}{2}-s_4}_{2,1}}\|\tilde{S}_{-4}g\|_{\dot{B}^{s_1+s_4}_{2,1}}.
\end{aligned}
$$

This completes the proof. □

We now introduce commutator estimates.

Lemma 3.10. *Let $s\in\left(-\frac{n}{2},\frac{n}{2}+1\right]$. There exists a sequence $c_j\in l^1(\mathbb{Z})$ such that $\|c_j\|_{l^1}=1$ and a constant C depending only on n and s such that*

$$
\forall j\in\mathbb{Z},\quad \|[f\cdot\nabla,\dot{\triangle}_j]g\|_{L^2}\le Cc_j2^{-sj}\|\nabla f\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|g\|_{\dot{B}^{s}_{2,1}}.
$$

See, e.g., [1] for a proof of Lemma 3.10.

4 Reformulation of the problem

In this section we first rewrite system (1) into the one for the perturbation. We then introduce some auxiliary lemmas which will be useful in the proof of the main result.

Let us rewrite the problem (1). We define μ_1, μ_2 and γ by

$$\mu_1 = \frac{\mu}{\bar\rho},\ \mu_2 = \frac{\mu+\mu'}{\bar\rho},\ \gamma = \sqrt{P'(\bar\rho)}.$$

By using the new unknown function

$$\sigma(t,x) = \frac{\rho(t,x)-\bar\rho}{\bar\rho}, \quad w(t,x) = \frac{1}{\gamma}u(t,x),$$

the initial value problem (1) is reformulated as

$$\begin{cases} \partial_t\sigma + \gamma\nabla\cdot w = F_1(U), \\ \partial_t w - \mu_1\Delta w - \mu_2\nabla(\nabla\cdot w) + \gamma\nabla\sigma = F_2(U), \\ (\sigma, w)(0,x) = (\sigma_0, w_0)(x), \end{cases} \tag{5}$$

where, $U = \begin{pmatrix} \sigma \\ w \end{pmatrix}$,

$$F_1(U) = -\gamma(w\cdot\nabla\sigma + \sigma\nabla\cdot w),$$

$$\begin{aligned} F_2(U) \;&=\; -\gamma(w\cdot\nabla)w - \mu_1\frac{\sigma}{\sigma+1}\Delta w - \mu_2\frac{\sigma}{\sigma+1}\nabla(\nabla\cdot w) \\ &\quad + \Big(\frac{\bar\rho\gamma}{\sigma+1} - \frac{\bar\rho}{\gamma}\frac{\int_0^1 P''(s\bar\rho\sigma+\bar\rho)ds}{\sigma+1}\Big)\sigma\nabla\sigma. \end{aligned}$$

We set

$$A = \begin{pmatrix} 0 & -\gamma\nabla\cdot \\ -\gamma\nabla & \mu_1\Delta + \mu_2\nabla\nabla\cdot \end{pmatrix}.$$

By using operator A, problem (5) is written as

$$\partial_t U - AU = F(U), \quad U|_{t=0} = U_0, \tag{6}$$

where

$$F(U) = \begin{pmatrix} F_1(U) \\ F_2(U) \end{pmatrix}, \quad U_0 = \begin{pmatrix} \sigma_0 \\ w_0 \end{pmatrix}.$$

We introduce a semigroup generated by A. We set

$$E(t)u := \mathfrak{F}^{-1}[e^{\hat A(\xi)t}\hat u] \quad \text{for } u \in L^2,$$

where

$$\hat A(\xi) = \begin{pmatrix} 0 & -i\gamma\xi^t \\ -i\gamma\xi & -\mu_1|\xi|^2 I_n - \mu_2\xi\xi^t \end{pmatrix}.$$

Here and in what follows the superscript $\cdot^t$ means the transposition.

5 Proof of main result

In this section we prove Theorem 2.1. In subsections 5.1 and 5.2 we establish the necessary estimates for $\triangle_{-1}U(t)$ and $\triangle_j U(t)$ for $j \geq 0$, respectively. In subsection 5.3 we derive the a priori estimate to complete the proof of Theorem 2.1.

We first explain known results which are used to prove Theorem 2.1.

Danchin [2] proved the following global existence result in nonhomogeneous Besov space.

Proposition 5.1 (Danchin [2]). *Let $n \geq 2$. There are two positive constants ϵ_1 and M such that for all (ρ_0, u_0) with $(\rho_0 - \bar{\rho}) \in \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}$, $u_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1}$ and*

$$\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \leq \epsilon_1, \tag{7}$$

problem (1) has a unique global solution $(\rho, u) \in C(\mathbb{R}^+; \dot{B}^{\frac{n}{2}}_{2,1} \cap \dot{B}^{\frac{n}{2}-1}_{2,1}) \times \big(L^1(\mathbb{R}^+; \dot{B}^{\frac{n}{2}+1}_{2,1}) \cap C(\mathbb{R}^+; \dot{B}^{\frac{n}{2}-1}_{2,1})\big)$ that satisfies the estimate

$$\sup_{t\geq 0}\{\|\rho(t)-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u(t)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\} + \int_0^\infty \|u\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} dt \leq M\big(\|\rho_0-\bar{\rho}\|_{\dot{B}^{\frac{n}{2}}_{2,1}\cap \dot{B}^{\frac{n}{2}-1}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\big).$$

Haspot [5] proved the following local existence result in nonhomogeneous Besov space.

Proposition 5.2 (Haspot [5]). *Let $n \geq 2$ and $1 \leq p < 2n$. Let $u_0 \in B^{\frac{n}{p}-1}_{p,1}$ and $(\rho_0 - \bar{\rho}) \in B^{\frac{n}{p}}_{p,1}$ with $\frac{1}{\rho_0}$ bounded away from zero. Then there exist a constant $T > 0$ such that the problem (1) has a local solution (ρ, u) on $[0, T]$ with $\frac{1}{\rho} > 0$ bounded away from zero and:*

$$\rho - \bar{\rho} \in C([0,T]; B^{\frac{n}{p}}_{p,1}), \quad u \in \big(C([0,T]; B^{\frac{n}{p}-1}_{p,1}) \cap L^1(0,T; B^{\frac{n}{p}+1}_{p,1})\big).$$

Moreover, this solution is unique if

$$p \leq n.$$

Proposition 5.3. *Let $T > 0$ and let (σ, w) be a solution of problem (6) on $[0, T]$ such that*

$$\sigma \in C([0,T]; B^{\frac{n}{2}}_{2,1}), w \in C([0,T]; B^{\frac{n}{2}}_{2,1}) \cap L^1(0,T; B^{\frac{n}{2}+1}_{2,1}), \tag{8}$$

Then, $\triangle_j U(t) = (\triangle_j \sigma, \triangle_j w)^{\mathrm{t}}$ for $j \geq -1$ satisfy

$$\partial_t \triangle_j U - A\triangle_j U = \triangle_j F(U), \tag{9}$$

$$\triangle_j U|_{t=0} = \triangle_j U_0. \tag{10}$$

Moreover, $\triangle_{-1}U(t)$ satisfy

$$\triangle_{-1}U(t) \in C\big([0,T]; \dot{B}^k_{2,1}\big), \quad \forall k \in [0,\infty) \tag{11}$$

and

$$\triangle_{-1}U(t) = E(t)\triangle_{-1}U_0 + \int_0^t E(t-s)\triangle_{-1}F(U)(s)ds. \tag{12}$$

Proof. Let $U(t) = (\sigma, w)^t$ be a solution of (6) satisfying (8). Since $\triangle_j AU = A\triangle_j U$, applying $\triangle_j$ to (6), we obtain (9) and (10). It then follows that

$$\triangle_j U(t) = E(t)\triangle_j U_0 + \int_0^t E(t-s)\triangle_j F(U)(s)ds.$$

We also have (11) from Lemma 3.1. This completes the proof. □

Set

$$\begin{aligned} M_1(t) \ := \ & \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|\triangle_{-1}U(\tau)\|_{L^2} \\ & + \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}}\sum_{j<0}2^j\|\dot{\triangle}_j U(\tau)\|_{L^2} \\ & + \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{2p}-\frac{1}{2}}\sum_{j<0}2^{(\frac{n}{2}-1)j}\|\dot{\triangle}_j U(\tau)\|_{L^2} \\ & + \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{2p}}\sum_{j<0}2^{\frac{n}{2}j}\|\dot{\triangle}_j U(\tau)\|_{L^2}, \end{aligned}$$

$$M_\infty(t) := \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{2p}}\sum_{j=0}^{\infty}2^{(\frac{n}{2}-1)j}\{\|\triangle_j U(\tau)\|_{L^2} + 2^j\|\triangle_j\sigma\|_{L^2}\},$$

$$M(t) := M_1(t) + M_\infty(t).$$

If we could obtain uniform estimates of $M_1(t)$ and $M_\infty(t)$, then Theorem 2.1 would be proved.

5.1 Estimate of low frequency parts

In this subsection we derive the estimate of $\triangle_{-1}U(t)$, in other words, we estimate $M_1(t)$.

Lemma 5.4. *(i) The set of all eigenvalues of $\hat{A}(\xi)$ consists of $\lambda_i(\xi)$ $(i = 1, 2, 3)$, where*

$$\begin{cases} \lambda_1(\xi) = \frac{-(\mu_1+\mu_2)|\xi|^2 + i|\xi|\sqrt{4\gamma^2-(\mu_1+\mu_2)|\xi|^2}}{2}, \\ \lambda_2(\xi) = \frac{-(\mu_1+\mu_2)|\xi|^2 - i|\xi|\sqrt{4\gamma^2-(\mu_1+\mu_2)|\xi|^2}}{2}, \\ \lambda_3(\xi) = -\mu_1|\xi|^2, \end{cases}$$

for all $\xi \in \mathbb{R}^n$.

(ii) $e^{t\hat{A}(\xi)}$ has the spectral resolution

$$e^{t\hat{A}(\xi)} = \sum_{j=1}^{3} e^{t\lambda_j(\xi)}P_j(\xi),$$

for all $|\xi| \neq \frac{2\gamma}{\sqrt{\mu_1+\mu_2}}$, where $P_j(\xi)$ is the eigenprojection for $\lambda_j(\xi)$.

For $|\xi| = \frac{2\gamma}{\sqrt{\mu_1+\mu_2}}$, *we have* $\lambda_1(\xi) = \lambda_2(\xi) = -\frac{\mu_1+\mu_2}{2}|\xi|^2$ *and*

$$e^{t\hat{A}(\xi)} = e^{t\lambda_1(\xi)}\big(I + t(\hat{A}(\xi) - \lambda_1 I)\big)P_1 + e^{t\lambda_3(\xi)}P_3$$

where $P_1(\xi), P_3(\xi)$ *is the eigenprojection for* $\lambda_1(\xi), \lambda_3(\xi)$.

Remark 5.5. *For each* $M > 0$ *there exist* $C_2 = C_2(M) > 0$ *and* $\beta_2 = \beta_2(M) > 0$ *such that the estimate*

$$\|e^{t\hat{A}(\xi)}\| \le C_2 e^{-\beta_2|\xi|^2 t}$$

holds for $|\xi| \le M$ *and* $t > 0$.

Lemma 5.6. *Let* $s \ge 0$. *Then* $E(t)$ *satisfies the estimates*

$$\|E(t)\Delta_{-1}U_0\|_{L^2} \le C(1+t)^{-\frac{n}{4}}\|\dot{S}_0 U_0\|_{\dot{B}^0_{1,\infty}},$$

$$\sum_{j<0} 2^{sj}\|E(t)\dot{\Delta}_j U_0\|_{L^2} \le C(1+t)^{-\frac{n}{4}-\frac{s}{2}}\|\dot{S}_0 U_0\|_{\dot{B}^0_{1,\infty}}$$

for $t \ge 0$.

To prove Lemma 5.6, we will use the following inequalities.

Lemma 5.7. *Let* $\alpha > 0$ *and* $s > -\frac{n}{2}$. *Then there holds the estimate*

$$\sum_{j<0}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}} |\xi|^{2s} e^{-2\alpha|\xi|^2 t} d\xi\Big)^{\frac{1}{2}} \le C(1+t)^{\frac{n}{4}-\frac{s}{2}}$$

for all $t > 0$.

We will prove Lemma 5.7 later. Now we prove Lemma 5.6.

Proof of Lemma 5.6. By Plancherel's theorem and Lemma 5.4 (*ii*), we have that there exists a constant $\beta' > 0$ such that

$$\begin{aligned}
\|E(t)\Delta_{-1}U_0(t)\|_{L^2} &\le C\Big(\int_{|\xi|\le 2} \big|e^{\hat{A}(\xi)t}\chi(\xi)\hat{U}_0(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}} \\
&\le C\sup_{j<0}\|\phi_j(\xi)\hat{U}_0\|_{L^\infty}\Big(\sum_{j<0}\int_{2^{j-1}|\xi|<2^{j+2}} e^{-2\beta'|\xi|^2 t} d\xi\Big)^{\frac{1}{2}} \\
&\le C(1+t)^{-\frac{n}{4}}\|\dot{S}_0 U_0\|_{\dot{B}^0_{1,\infty}}, \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j<0} 2^{sj}\|E(t)\dot\Delta_j U_0(t)\|_{L^2} &\le C\sum_{j<0} 2^{sj}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}} \big|e^{\hat A(\xi)t}\phi_j(\xi)\hat{U_0}(\xi)\big|^2 d\xi\Big)^{\frac12}\\
&\le C\sum_{j<0}\Big(\int_{2^{j-1}<|\xi|\le 2^{j+2}} |\xi|^{2s}e^{-2\beta'|\xi|^2 t}|\phi_j(\xi)\hat{U_0}(\xi)|^2 d\xi\Big)^{\frac12}\\
&\le C\sum_{j<0}\|\dot\Delta_j U_0\|_{L^1}\Big(\int_{2^{j-1}<|\xi|\le 2^{j+2}} |\xi|^{2s}e^{-2\beta'|\xi|^2 t}d\xi\Big)^{\frac12}\\
&\le C(1+t)^{-\frac n4-\frac s2}\|\dot S_0 U_0\|_{\dot B^0_{1,\infty}}. \qquad (14)
\end{aligned}$$

Here we used Lemma 5.7.

The desired estimates of Lemma 5.6 follow from (13) and (14). □

It remains to prove Lemma 5.7.

Proof of Lemma 5.7. Let $\alpha > 0$ and $s > -\frac n2$. We have

$$\begin{aligned}
&\sum_{j<0}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}} |\xi|^{2s}e^{-2\alpha|\xi|^2 t}d\xi\Big)^{\frac12}\\
\le\ & C\sum_{j<0} 2^{js}\Big(\int_{|\xi|<2^{j+2}} d\xi\Big)^{\frac12}\\
\le\ & C\sum_{j<0} 2^{j(s+\frac n2)} \le C. \qquad (15)
\end{aligned}$$

We will next show the the inequality

$$\sum_{j<0}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}} |\xi|^{2s}e^{-2\alpha|\xi|^2 t}d\xi\Big)^{\frac12} \le Ct^{-\frac n4-\frac s2}. \qquad (16)$$

By the substitution $\eta = t^{\frac12}\xi$, we obtain

$$\begin{aligned}
&\sum_{j<0}\Big(\int_{2^{j-1}<|\xi|<2^{j+2}} |\xi|^{2s}e^{-2\alpha|\xi|^2 t}d\xi\Big)^{\frac12}\\
=\ & t^{-\frac n4-\frac s2}\sum_{j<0}\Big(\int_{2^{j-1}\sqrt t<|\xi|<2^{j+2}\sqrt t} |\eta|^{2s}e^{-2\alpha|\eta|^2}d\xi\Big)^{\frac12}.
\end{aligned}$$

If $t \le 1$, we can easily prove (16).

We suppose $t > 1$. There exist an integer $J < 0$ such that $2^{-2J} < t < 2^{-2(J-1)}$.

We have

$$
\begin{aligned}
&\sum_{j<0}\Big(\int_{2^{j-1}\sqrt{t}<|\xi|<2^{j+2}\sqrt{t}}|\eta|^{2s}e^{-2\alpha|\eta|^2}d\xi\Big)^{\frac{1}{2}}\\
\leq\ &\sum_{j\leq J}\Big(\int_{2^{j-J-1}<|\xi|<2^{j-J+3}}|\eta|^{2s}e^{-2\alpha|\eta|^2}d\xi\Big)^{\frac{1}{p_0}}\\
&+\sum_{J<j<0}\Big(\int_{2^{j-J-1}<|\xi|<2^{j-J+3}}|\eta|^{2s}e^{-2\alpha|\eta|^2}d\xi\Big)^{\frac{1}{2}}\\
=:\ & I_1+I_2.
\end{aligned}
$$

By the substitution $k=j-J$, we have

$$
I_1=\sum_{k\leq 0}\Big(\int_{2^{k-1}<|\xi|<2^{k+3}}|\eta|^{2s}e^{-p_0\alpha|\eta|^2}d\xi\Big)^{\frac{1}{p_0}}<C,
$$

and

$$
\begin{aligned}
I_2\ &\leq\ \sum_{k>0}\Big(\int_{2^{k-1}<|\xi|<2^{k+3}}|\eta|^{2s}e^{-2\alpha|\eta|^2}d\xi\Big)^{\frac{1}{2}}\\
&\leq\ C\sum_{k>0}e^{-\frac{1}{2}2^k}\Big(\int_{2^{k-1}<|\xi|<2^{k+3}}|\eta|^{2s}e^{-\alpha|\eta|^2}d\xi\Big)^{\frac{1}{2}}\\
&\leq\ C\sum_{k>0}e^{-\frac{1}{2}2^k}\leq C.
\end{aligned}
$$

Hence we obtain (16). By (15) and (16) we have the desired inequality . $\square$

As for $M_1(t)$, we show the following estimate.

Proposition 5.8. *There exists a constant $C>0$ independent of T such that*

$$
M_1(t)\leq C\|U_0\|_{\dot{B}^0_{1,\infty}}+CM(t)\int_0^t\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau+CM^2(t)
$$

for $t\in[0,T]$.

To prove Proposition 5.8, we will use the following estimate on $F(U)$.

Lemma 5.9. *There exists a constant $C>0$ independent of T such that*

$$
\|\dot{S}_0F(U)\|_{\dot{B}^0_{1,\infty}}\ \leq\ C(1+t)^{-\frac{n}{2}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}+C(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t)
$$

for $t\in[0,T]$.

We will prove Lemma 5.9 later. Now we prove Proposition 5.8.

Proof of Proposition 5.8. By Lemma 5.6 and (12), we see that

$$\begin{aligned}\|\Delta_{-1}U(\tau)\|_{L^2} &\leq \|E(\tau)\Delta_{-1}U_0\|_{L^2} + \int_0^\tau \|E(\tau-\tau')\Delta_{-1}F(U(\tau'))\|_{L^2}d\tau' \\ &\leq C(1+\tau)^{-\frac{n}{4}}\|\dot{S}_0U_0\|_{\dot{B}^0_{1,\infty}} \\ &\quad + \int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}}\|\dot{S}_0F(U(\tau'))\|_{\dot{B}^0_{1,\infty}}ds, \qquad (17)\end{aligned}$$

and

$$\begin{aligned}\sum_{j<0} 2^{sj}\|\dot{\Delta}_jU(\tau)\|_{L^2} &\leq \sum_{j<0}\|E(\tau)\dot{\Delta}_jU_0\|_{L^2} + \int_0^\tau \sum_{j<0}\|E(\tau-\tau')\dot{\Delta}_jF(U(\tau'))\|_{L^2}d\tau' \\ &\leq C(1+\tau)^{-\frac{n}{4}-\frac{s}{2}}\|\dot{S}_0U_0\|_{\dot{B}^0_{1,\infty}} \\ &\quad + \int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}\|\dot{S}_0F(U(\tau'))\|_{\dot{B}^0_{1,\infty}}d\tau' \qquad (18)\end{aligned}$$

for $s>0$.

Using Lemma 5.9, for $0\leq s\leq \frac{n}{2}$, we have

$$\begin{aligned}&\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}\|\dot{S}_0F(U(\tau'))\|_{\dot{B}^0_{1,\infty}}d\tau' \\ \leq\ & C\int_0^t (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}\{(1+\tau')^{-\frac{n}{2}}M(\tau')\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + (1+\tau')^{-\frac{n}{2}-\frac{1}{2}}M^2(\tau')\}d\tau' \\ \leq\ & CM(t)\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}(1+\tau')^{-\frac{n}{2}}\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' \\ & +CM^2(t)\int_0^\tau (1+\tau-\tau')^{-\frac{n}{4}-\frac{s}{2}}(1+\tau')^{-\frac{n}{2}-\frac{1}{2}}d\tau' \\ \leq\ & C(1+\tau)^{-\frac{n}{4}-\frac{s}{2}}M(t)\int_0^\tau \|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' + C(1+\tau)^{-\frac{n}{4}-\frac{s}{2}}M^2(t). \qquad (19)\end{aligned}$$

Here we used Lemma 3.8 and the facts that $\frac{n}{2}+\frac{1}{2}>1$ for $n\geq 2$. By (17) and (19), we obtain

$$\begin{aligned}\|\Delta_{-1}U(\tau)\|_{L^2} &\leq C(1+\tau)^{-\frac{n}{4}}\|U_0\|_{\dot{B}^0_{1,\infty}} \\ &\quad +C(1+\tau)^{-\frac{n}{4}}M(t)\int_0^t \|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' + C(1+\tau)^{-\frac{n}{4}}M^2(t),\end{aligned}$$

and hence,

$$(1+\tau)^{\frac{n}{4}}\|\Delta_{-1}U(\tau)\|_2 \leq C\|U_0\|_{\dot{B}^0_{1,\infty}} + CM(t)\int_0^t \|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' + CM^2(t).$$

Similarly, we get estimates

$$(1+\tau)^{\frac{n}{4}+\frac{1}{2}}\sum_{j<0}2^j\|\dot{\Delta}_jU(\tau)\|_2 \leq C\|U_0\|_{\dot{B}^0_{1,\infty}} + CM(t)\int_0^t \|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau' + CM^2(t),$$

$$(1+\tau)^{\frac{n}{2}-\frac{1}{2}}\sum_{j<0}2^{(\frac{n}{2}-1)j}\|\dot{\triangle}_jU(\tau)\|_2\le C\|U_0\|_{\dot{B}^0_{1,\infty}}+CM(t)\int_0^t\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t),$$

$$(1+\tau)^{\frac{n}{2}}\sum_{j<0}2^{\frac{n}{2}j}\|\dot{\triangle}_jU(\tau)\|_2\le C\|U_0\|_{\dot{B}^0_{1,\infty}}+CM(t)\int_0^t\|w(\tau')\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau'+CM^2(t).$$

Taking the supremum in $\tau\in[0,t]$, we obtain the desired estimate. □

It remains to prove Lemma 5.9.

Proof of Lemma 5.9. We consider each term of $F(U)$. By Lemma 3.9, we have

$$\begin{aligned}\sup_{j<0}\|\dot{\triangle}_j(w\cdot\nabla\sigma)\|_{L^1} &\le C\{\|\dot{S}_4w\|_{L^2}\|\dot{S}_4\nabla\sigma\|_{L^2}+\|\tilde{S}_0w\|_{L^2}\|\tilde{S}_0\nabla\sigma\|_{L^2}\}\\ &\le C(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t),\end{aligned}$$

$$\begin{aligned}\sup_{j<0}\|\dot{\triangle}_j(\sigma\nabla\cdot w)\|_{L^1} &\le C\{\|\dot{S}_4\sigma\|_{L^2}\|\dot{S}_4\nabla w\|_{L^2}+\|\tilde{S}_0\sigma\|_{L^2}\|\tilde{S}_0\nabla w\|_{L^2}\}\\ &\le C\{\|\dot{S}_4\sigma\|_{L^2}(\|\dot{S}_0\nabla w\|_{L^2}+\|\dot{\triangle}_0w\|_{L^2}+\|\dot{\triangle}_1w\|_{L^2}\\ &\quad+\|\dot{\triangle}_2w\|_{L^2}+\|\dot{\triangle}_3w\|_{L^2})+\|\tilde{S}_0\sigma\|_{L^2}\|\tilde{S}_0\nabla w\|_{L^2}\}\\ &\le C\{(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.\end{aligned}$$

Similarly, we have

$$\sup_{j<0}\|\dot{\triangle}_j(w\cdot\nabla w)\|_{L^1}\le C\{(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.$$

We obtain by Lemma 3.1, 3.7 and 3.9

$$\begin{aligned}\sup_{j<0}\|\dot{\triangle}_j(\frac{\sigma}{\sigma+1}\triangle w)\|_{L^1} &\le C\{\|\dot{S}_4(\frac{\sigma}{\sigma+1})\|_{L^2}\|\dot{S}_4\triangle w\|_{L^2}+\|\tilde{S}_0(\frac{\sigma}{\sigma+1})\|_{L^2}\|\tilde{S}_0\triangle w\|_{L^2}\}\\ &\le C\{\|\sigma\|_{L^2}\|\dot{S}_4w\|_{\dot{B}^1_{2,1}}++\|\tilde{S}_0(\frac{\sigma}{\sigma+1})\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\tilde{S}_0\triangle w\|_{L^2}\}\\ &\le C\{(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t)+(1+t)^{-\frac{n}{2}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}.\end{aligned}$$

The other terms are estimated similarly, and we arrive at

$$\sup_{j<0}\|\dot{\triangle}F(U)\|_{L^1}\le C(1+t)^{-\frac{n}{2}-\frac{1}{2}}M^2(t)+C(1+t)^{-\frac{n}{2}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.$$

This completes the proof. □

5.2 Estimate of high frequency parts

We next derive estimates for $M_\infty(t)$. The system (9) is written as

$$\begin{cases}\partial_t\triangle_j\sigma+\gamma\nabla\cdot\triangle_jw=\triangle_jF_1(U),\\ \partial_t\triangle_jw-\mu_1\triangle\triangle_jw-\mu_2\nabla\cdot(\nabla\triangle_jw)+\gamma\nabla\triangle_j\sigma=\triangle_jF_2(U).\end{cases}\tag{20}$$

Proposition 5.10. *Let $j \geq 0$. There holds*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\triangle_j U(t)\|_{L^2}^2 + \mu_1\|\nabla\triangle_j w(t)\|_{L^2}^2 + \mu_2\|\nabla\cdot\triangle_j w(t)\|_{L^2}^2\\ = \ &(\triangle_j F_1(U), \triangle_j\sigma) + (\triangle_j F_2(U), \triangle_j w)\end{aligned} \tag{21}$$

for a.e. $t \in [0,T]$.

See, e.g., [12], for the proof of Lemma 5.10.

We recall that for $s \in \mathbb{R}$, Λ^s is defined by $\Lambda^s z := \mathfrak{F}^{-1}[|\xi|^s\hat{z}]$. Let $d = \Lambda^{-1}\nabla\cdot w$ be the "compressible part" of the velocity. Applying $\Lambda^{-1}\nabla\cdot$ to $(20)_2$, system (20) writes

$$\begin{cases}\partial_t\triangle_j\sigma + \gamma\Lambda\triangle_j d = \triangle_j F_1(U),\\ \partial_t\triangle_j d - \nu\triangle\triangle_j d - \gamma\Lambda\triangle_j\sigma = \Lambda^{-1}\nabla\cdot\triangle_j F_2(U),\end{cases} \tag{22}$$

where we denote $\nu = \mu_1 + \mu_2$.

Proposition 5.11. *Let $j \geq 0$. There holds*

$$\begin{aligned}&\frac{1}{2}\frac{\nu}{\gamma}\frac{d}{dt}\|\Lambda\triangle_j\sigma\|_{L^2}^2 - \frac{d}{dt}(\Lambda\triangle_j\sigma, \triangle_j d) + \|\Lambda\triangle_j\sigma\|_{L^2}^2 = \gamma\|\Lambda\triangle_j d\|_{L^2}^2\\ &-(\Lambda\triangle_j F_1(U), \triangle_j d) - (\Lambda^{-1}\nabla\cdot\triangle_j F_2(U), \Lambda\triangle_j\sigma) + \frac{\nu}{\gamma}(\Lambda\triangle_j F_1(U), \Lambda\triangle_j\sigma)\end{aligned} \tag{23}$$

for a.e. $t \in [0,T]$.

See, e.g., [12], for the proof of Lemma 5.11.
We introduce a lemma for estimates of the right-hand side of (23).

Lemma 5.12. *The following inequalities hold*

(*i*)

$$\left|\left(\Lambda\triangle_j(w\cdot\nabla\sigma), \Lambda\triangle_j\sigma\right)\right| \leq C\alpha_j 2^{-(\frac{n}{2}-1)j}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\Lambda\triangle_j\sigma\|_{L^2},$$

(*ii*)

$$\begin{aligned}&\left|\left(\Lambda\triangle_j(w\cdot\nabla\sigma), \triangle_j d\right)\right|\\ \leq \ &C\big\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2}\\ &+\|\nabla\triangle_j\sigma\|_{L^2}\big(2^j\|\dot{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} + 2^{2j}\|\tilde{S}_0 w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\triangle_j d\|_{L^2}\big)\big\},\end{aligned}$$

where C is independ of $j \in \mathbb{Z}$ and $\{\alpha_j\}$ with $\|\{\alpha_j\}\|_{l^1} \leq 1$.

Proof. As for (*i*), see, e.g., [2].

Let us prove (ii). By using Lemma 3.10, we obtain

$$\begin{aligned}
&\big|\big(\Lambda\triangle_j(w\cdot\nabla\sigma),\triangle_j d\big)\big| \\
\le\ &\big|\big([w\cdot\nabla,\triangle_j]\sigma,\Lambda\triangle_j d\big)\big|+\big|\big(w\cdot\nabla\triangle_j\sigma,\Lambda\triangle_j d\big)\big| \\
\le\ &C\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|\nabla w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} \\
&+\|\nabla\triangle_j\sigma\|_{L^2}\big(\|\dot S_0 w\|_{L^\infty}\|\Lambda\triangle_j d\|_{L^2}+\|\tilde S_0 w\|_{L^n}\|\Lambda\triangle_j d\|_{L^{\frac{2n}{n-2}}}\big)\} \\
\le\ &C\big\{\alpha_j 2^{-(\frac{n}{2}-1)j}\|\nabla w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2} \\
&+\|\nabla\triangle_j\sigma\|_{L^2}\big(2^j\|\dot S_0 w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\triangle_j d\|_{L^2}+2^{2j}\|\tilde S_0 w\|_{\dot B^{\frac{n}{2}-1}_{2,1}}\|\triangle_j d\|_{L^2}\big)\big\}.
\end{aligned}$$

This completes the proof. □

Proposition 5.13. *There holds*

$$\begin{aligned}
&\frac{d}{dt}E_j(t)+c_0E_j(t) \\
\le\ &C\{\alpha_j(1+t)^{-\frac{n}{2}}M(t)\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}+(1+t)^{-\frac{n}{2}}2^{(\frac{n}{2}+1)j}\|\triangle_j d\|_{L^2}M(t) \\
&+2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2}+2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2} \\
&+2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}\},
\end{aligned}\tag{24}$$

for $t\in[0,T]$ and $j\ge 1$, where $\sum_{j\in\mathbb{Z}}\alpha_j\le 1$, and c_0 is a positive constant independent of j. Here, $E_j(t)$ is equivalent to $2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\triangle_j\sigma(t)\|_{L^2}$. That is, there exists a positive constant D_1 such that

$$\begin{aligned}
&\frac{1}{D_1}\big(2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\triangle_j\sigma(t)\|_{L^2}\big) \\
\le\ &E_j(t) \\
\le\ &D_1\big(2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\triangle_j\sigma(t)\|_{L^2}\big).
\end{aligned}$$

Proof. We add (21) to $\kappa\times$ (23) with a constant $\kappa>0$ to be determined later. Then, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{2}\|\triangle_j U\|_{L^2}^2+\frac{\kappa}{2}\frac{\nu}{\gamma}\|\Lambda\triangle_j\sigma\|_{L^2}^2-\kappa(\Lambda\triangle_j\sigma,\triangle_j d)\Big\} \\
&+\mu_1\|\nabla\triangle_j w\|_{L^2}^2+\mu_2\|\nabla\cdot\triangle_j w\|_{L^2}^2+\kappa\|\Lambda\triangle_j\sigma\|_{L^2}^2 \\
=\ &\gamma\kappa\|\Lambda\triangle_j w\|_{L^2}^2+\big(\triangle_j F_1(U),\triangle_j\sigma\big)+\big(\triangle_j F_2(U),\triangle_j w\big)+\kappa\frac{\nu}{\gamma}\big(\Lambda\triangle_j F_1(U),\Lambda\triangle_j\sigma\big) \\
&-\kappa\big(\Lambda\triangle_j F_1(U),\triangle_j d\big)-\kappa\big(\Lambda^{-1}\nabla\cdot\triangle_j F_2(U),\Lambda\triangle_j\sigma\big).
\end{aligned}\tag{25}$$

We set

$$E_j^2(t)=2^{2(\frac{n}{2}-1)j}\Big\{\frac{1}{2}\|\triangle_j U\|_{L^2}^2+\frac{\kappa}{2}\frac{\nu}{\gamma}\|\Lambda\triangle_j\sigma\|_{L^2}^2-\kappa(\Lambda\triangle_j\sigma,\triangle_j d)\Big\}.$$

For each $\kappa\le 1$, there exists a $D_1>\max\{3,\frac{1}{\kappa},\frac{\mu}{8\gamma}\}$ such that

$$E_j^2\le D_1^2\big(2^{(\frac{n}{2}-1)j}\|\triangle_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\triangle_j\sigma\|_{L^2}\big)^2.$$

By Cauchy's inequality with δ, we have

$$\begin{aligned}
&\big(2^{(\frac{n}{2}-1)j}\|\Delta_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\Delta_j\sigma\|_{L^2}\big)^2+D_1^2\kappa 2^{2(\frac{n}{2}-1)j}(\Delta_j d,\Lambda\Delta_j\sigma)\\
\le\ & 2\big\{\big(2^{(\frac{n}{2}-1)j}\|\Delta_j U(t)\|_{L^2}\big)^2+\big(2^{\frac{n}{2}j}\|\Delta_j\sigma\|_{L^2}\big)^2\big\}\\
&+D_1^2\kappa\delta\big(2^{(\frac{n}{2}-1)j}\|\Lambda\Delta_j\sigma\|_{L^2}\big)^2+D_1^2\kappa\frac{1}{4\delta}\big(2^{(\frac{n}{2}-1)j}\|\Delta_j w\|_{L^2}\big)^2.
\end{aligned}$$

We select $\delta=\frac{\nu}{4\gamma D_1}$ and κ is fixed in such a way that $\kappa\le\min\{\frac{\mu}{4\gamma},1\}$. We then obtain

$$\begin{aligned}
&\frac{1}{D_1^2}\big(2^{(\frac{n}{2}-1)j}\|\Delta_j U(t)\|_{L^2}+2^{\frac{n}{2}j}\|\Delta_j\sigma\|_{L^2}\big)^2\\
\le\ & 2^{2(\frac{n}{2}-1)j}\Big\{\frac{1}{2}\|\Delta_j U\|_{L^2}^2+\frac{\kappa\,\nu}{2\,\gamma}\|\Lambda\Delta_j\sigma\|_{L^2}^2-\kappa(\Lambda\Delta_j\sigma,\Delta_j d)\Big\}=E_j^2.
\end{aligned}$$

For $j\ge 0$, by Lemma 3.1, and that there exists a $c_0>0$ such that

$$2c_0E_j^2\le 2^{2(\frac{n}{2}-1)j}\big\{\mu_1\|\nabla\Delta_j w\|_{L^2}^2+\mu_1\|\nabla\cdot\Delta_j w\|_{L^2}^2+\kappa\|\Lambda\Delta_j\sigma\|_{L^2}^2-\gamma\kappa\|\Lambda\Delta_j w\|_{L^2}^2\big\}.$$

Let us next estimate the right-hand side of $2^{2(\frac{n}{2}-1)j}\times(25)$. By Hölder's inequality, we obtain

$$\begin{aligned}
2^{2(\frac{n}{2}-1)j}\big(\Delta_j F_1(U),\Delta_j\sigma\big)&\le 2^{(\frac{n}{2}-1)j}\|\Delta_j F_1(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\Delta_j\sigma\|_{L^2},\\
2^{2(\frac{n}{2}-1)j}\big(\Delta_j F_2(U),\Delta_j w\big)&\le 2^{(\frac{n}{2}-1)j}\|\Delta_j F_2(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\Delta_j w\|_{L^2},\\
2^{2(\frac{n}{2}-1)j}\big(\Lambda^{-1}\nabla\cdot\Delta_j F_2(U),\Delta_j\sigma\big)&\le 2^{(\frac{n}{2}-1)j}\|\Delta_j F_2(U)\|_{L^2}2^{(\frac{n}{2}-1)j}\|\Delta_j\sigma\|_{L^2}.
\end{aligned}$$

By Lemma 5.12 we have

$$\begin{aligned}
&2^{2(\frac{n}{2}-1)j}\big(\Lambda\Delta_j F_1(U),\Lambda\Delta_j\sigma\big)\\
=\ & 2^{2(\frac{n}{2}-1)j}\big(\Lambda\Delta_j(w\cdot\nabla\sigma),\Lambda\Delta_j\sigma\big)+2^{2(\frac{n}{2}-1)j}\big(\Lambda\Delta_j(\sigma\nabla\cdot w),\Lambda\Delta_j\sigma\big)\\
\le\ & C\alpha_j\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}2^{(\frac{n}{2}-1)j}\|\Lambda\Delta_j\sigma\|_{L^2}+2^{2(\frac{n}{2}-1)j}\|\Lambda\Delta_j(\sigma\nabla\cdot w)\|_{L^2}\|\Lambda\Delta_j\sigma\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
&2^{2(\frac{n}{2}-1)j}\big(\Lambda\Delta_j F_1(U),\Delta_j d\big)\\
=\ & 2^{2(\frac{n}{2}-1)j}\big(\Lambda\Delta_j(w\cdot\nabla\sigma),\Delta_j d\big)+2^{2(\frac{n}{2}-1)j}\big(\Lambda\Delta_j(\sigma\nabla\cdot w),\Delta_j d\big)\\
\le\ & C\big\{\alpha_j2^{-(\frac{n}{2}-1)j}\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}\|\sigma\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\Delta_j d\|_{L^2}\\
&+\|\nabla\Delta_j\sigma\|_{L^2}\big(2^j\|\dot S_0 w\|_{\dot B^{\frac{n}{2}}_{2,1}}\|\Delta_j d\|_{L^2}+2^{2j}\|\tilde S_0 w\|_{\dot B^{\frac{n}{2}-1}_{2,1}}\|\Delta_j d\|_{L^2}\big)\big\}\\
&+2^{2(\frac{n}{2}-1)j}\|\Lambda\Delta_j(\sigma\nabla\cdot w)\|_{L^2}\|\Delta_j d\|_{L^2},
\end{aligned}$$

where $\sum_{j\in\mathbb{Z}}\alpha_j\le 1$. Hence we obtain

$$\begin{aligned}
&\frac{d}{dt}E_j^2+2c_0E_j^2\le CE_j\big\{\alpha_j(1+t)^{-\frac{n}{2}}M(t)\|w\|_{\dot B^{\frac{n}{2}+1}_{2,1}}\\
&+(1+t)^{-\frac{n}{2}}2^{(\frac{n}{2}+1)j}\|\Delta_j d\|_{L^2}M(t)+2^{(\frac{n}{2}-1)j}\|\Lambda\Delta_j(\sigma\nabla\cdot w)\|_{L^2}\\
&+2^{(\frac{n}{2}-1)j}\|\Delta_j F_1(U)\|_{L^2}+2^{(\frac{n}{2}-1)j}\|\Delta_j F_2(U)\|_{L^2}\big\}. \qquad (26)
\end{aligned}$$

From (26) and dividing by E_j, we get the desired result. $\square$

5.3 Proof of Theorem 2.1.

Proposition 5.14. *There exists a constant $\epsilon_2 > 0$ such that if*

$$\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap \dot{B}^0_{1,\infty}} + \|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \le \epsilon_2,$$

then there holds

$$M(t) \le C\{\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap \dot{B}^0_{1,\infty}} + \|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\}$$

for $0 \le t \le T$, where the constant C does not depend on T.

Proof. By (24) we have

$$\begin{aligned} E_j(t) \;&\le\; e^{-c_0 t}E_j(0) \\ &+C\int_0^t e^{-c_0(t-\tau)}\{\alpha_j(1+\tau)^{-\frac{n}{2}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} \\ &+(1+\tau)^{-\frac{n}{2}}2^{(\frac{n}{2}+1)j}\|\triangle_j d\|_{L^2}M(t) \\ &+2^{(\frac{n}{2}-1)j}\|\Lambda\triangle_j(\sigma\nabla\cdot w)\|_{L^2} \\ &+2^{(\frac{n}{2}-1)j}\|\triangle_j F_1(U)\|_{L^2} + 2^{(\frac{n}{2}-1)j}\|\triangle_j F_2(U)\|_{L^2}\}d\tau, \end{aligned} \tag{27}$$

where $\sum_{j=0}^{\infty}\alpha_j \le 1$. Hence summing up on $j \ge 0$, by the monotone convergence theorem, we obtain

$$\begin{aligned} &\sum_{j=0}^{\infty}E_j(t) \le e^{-c_0 t}\sum_{j=0}^{\infty}E_j(0) \\ &+C\int_0^t e^{-c_0(t-\tau)}\{(1+\tau)^{-\frac{n}{2}}M(t)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \sum_{j=0}^{\infty}2^{j\frac{n}{2}}\|\dot{\triangle}_j(\sigma\nabla\cdot w)\|_{L^2} \\ &+\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_1(U)\|_{L^2} + \sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_2(U)\|_{L^2}\}d\tau. \end{aligned} \tag{28}$$

We next estimate the right-hand side of (28). From Lemma 3.6, we have

$$\sum_{j=0}^{\infty}2^{j\frac{n}{2}}\|\dot{\triangle}_j\sigma\nabla\cdot w\|_{L^2} \le \|\sigma\nabla\cdot w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \le C\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla\cdot w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \le C(1+\tau)^{-\frac{n}{2}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.$$

Let us next consider the quantities $\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j F_1(U)\|_{L^2}$:

$$\begin{aligned} \sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\triangle}_j(w\cdot\nabla\sigma)\|_{L^2} \;&\le\; \|w\cdot\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\ &\le\; C\|w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \\ &\le\; C(\|\dot{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}} + \|\tilde{S}_0 w\|_{\dot{B}^{\frac{n}{2}}_{2,1}})\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}} \\ &\le\; C(1+\tau)^{-n}M^2(\tau) + C(1+\tau)^{-\frac{n}{2}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}, \end{aligned}$$

$$\begin{aligned}\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\Delta}_j(\sigma\nabla\cdot w)\|_{L^2} &\leq \|\sigma\nabla\cdot w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\leq C\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\leq C(1+\tau)^{-n}M^2(\tau)+C(1+\tau)^{-\frac{n}{2}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.\end{aligned}$$

Hence, we obtain the estimate of $\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\Delta}_jF_1(U)\|_{L^2}$. By using Lemma 3.6, Lemma 3.7 and Lemma 3.9, $\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\Delta}_jF_2(U)\|_{L^2}$ is estimated as

$$\begin{aligned}\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\Delta}_j(w\cdot\nabla)w\| &\leq C\{\|\dot{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\tilde{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\quad+\|\dot{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}+\|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|\tilde{S}_{-5}\nabla w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\}\\ &\leq C(1+\tau)^{-\frac{n}{2}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.\end{aligned}$$

Here we used

$$\|\tilde{S}_{-5}w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\leq C\{(\sum_{j=-5}^{-1}2^{j\frac{n}{2}}\|\dot{\Delta}_jw\|_{L^2})+\|\tilde{S}_0w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\}\leq C(1+\tau)^{-\frac{n}{2}}M(\tau),$$

$$\|\tilde{S}_{-4}w\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\leq C\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}.$$

$$\begin{aligned}\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\Delta}_j(\frac{\sigma}{\sigma+1}\Delta w)\|_{L^2} &\leq \|\frac{\sigma}{\sigma+1}\Delta w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\leq C\|\frac{\sigma}{\sigma+1}\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\Delta w\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\leq C\|\sigma\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\\ &\leq C(1+\tau)^{-\frac{n}{2}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}},\end{aligned}$$

$$\begin{aligned}\sum_{j=0}^{\infty}2^{j(\frac{n}{2}-1)}\|\dot{\Delta}_j(\frac{\sigma}{\sigma+1}\nabla\sigma)\|_{L^2} &\leq \|\frac{\sigma}{\sigma+1}\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\leq C\|\frac{\sigma}{\sigma+1}\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|\nabla\sigma\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\\ &\leq C(1+\tau)^{-n}M^2(\tau).\end{aligned}$$

In the same way as above, we can obtain estimates of other terms on $\|F_2(U)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}$. Hence, by using Lemma 3.8, the integral of the right-hand side of (28) is estimated as

$$\begin{aligned}&\int_0^t e^{-c_0(t-\tau)}\{(1+\tau)^{-n}M^2(\tau)+(1+\tau)^{-\frac{n}{2}}M(\tau)\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\}d\tau\\ \leq\;& M(t)\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{n}{2}}\|w\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau+M^2(t)\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-n}d\tau\\ \leq\;& C(1+t)^{-\frac{n}{2}}\epsilon_2M(t)+C(1+t)^{-n}M^2(t).\end{aligned}$$

Hence, we obtain

$$M_\infty(t) \le C\big(\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big) + C\epsilon_2 M(t) + CM^2(t). \tag{29}$$

By Proposition 5.8 and (29), we have

$$M(t) \le C\big(\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap \dot{B}^0_{1,\infty}} + \|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big) + C\epsilon_2 M(t) + CM^2(t).$$

By taking $\epsilon_2 > 0$ suitably small, we obtain

$$M(t) \le C\big(\|U_0\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}\cap \dot{B}^0_{1,\infty}} + \|\sigma_0\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\big)$$

for all $0 \le t \le T$ with C independent of T. This completes the proof. □

It follows from Proposition 5.2 and Proposition 5.14 that

$$M(t) \le C_3 \qquad \text{for all } t,$$

if the initial perturbation is sufficiently small. Hence we obtain the desired decay estimate (2.1), (2.1) and (2.1) of Theorem 2.1.

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